Relativistic Corrections in the Theory of Muon Capture*

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(Received 21 May 1963; revised manuscript received 18 September 1963)

Computations are presented in closure approximation on the effect of relativistic corrections (∼[velocity of capturing proton]/[velocity of light]) on calculated muon total capture rates. A numerical estimate is made for the cases $\mu^- + {}_2He^3 \rightarrow {}_1H^3 + \nu_\mu$ and $\mu^- + {}_2He^4 \rightarrow {}_1H^4 + \nu_\mu$ and a comparison with experiment is given.

I. INTRODUCTION

In view of the increasing availability of high-intensity muon beams, experiments are now technically possible on muon capture by very light nuclei, viz.: ${}_1H^2$, ${}_2He^3$, ${}_2He^4$, and some results have already been reported.[1] Because of the importance of the analysis of such experiments for the determination of the underlying muon-nucleon weak interaction, we compute in this paper relativistic corrections to the muon capture matrix elements which are of order [velocity of capturing proton]/[velocity of light] and which have not been included in previous calculations of muon total capture rates.

II. GENERAL EXPRESSION FOR MUON CAPTURE MATRIX ELEMENTS

We begin with Eqs. (2a)–(3d) of P[2] and use for $|\mathrm{M.E.}_{\mathrm{nucl}}{}^{(\mu)}(a\rightarrow b)|^2$ the expression of Eq. (2b) of P augmented by the relativistic correction terms $\sim|\mathbf{p}_i/m_p|$ given in Eq. (5b) of FP.[3] We then obtain for $\Lambda^{(\mu)}(a)$, the muon total capture rate of the parent nucleus in the state $|a\rangle$,

$$\begin{aligned}\Lambda^{(\mu)}(a)=&\frac{Z^3}{2\pi^2}\frac{m_\mu{}^5}{(137)^3}\sum_b(\eta_{ba})^2\int\frac{d\hat{\nu}}{4\pi}\{(G_V{}^{(\mu)})^2|\langle b|\sum_i\tau_i{}^{(-)}\exp(-i\boldsymbol{\nu}_{ba}\cdot\mathbf{r}_i)\varphi(r_i)|a\rangle|^2\\&+(G_A{}^{(\mu)})^2|\langle b|\textstyle\sum_i\tau_i{}^{(-)}\exp(-i\boldsymbol{\nu}_{ba}\cdot\mathbf{r}_i)\varphi(r_i)\boldsymbol{\sigma}_i|a\rangle|^2\\&+[(G_P{}^{(\mu)})^2-2G_A{}^{(\mu)}G_P{}^{(\mu)}]|\langle b|\textstyle\sum_i\tau_i{}^{(-)}\exp(-i\boldsymbol{\nu}_{ba}\cdot\mathbf{r}_i)\varphi(r_i)\boldsymbol{\sigma}_i|a\rangle\cdot\hat{\nu}|^2\\&-(G_V{}^{(\mu)}g_V{}^{(\mu)})[\langle b|\textstyle\sum_i\tau_i{}^{(-)}\exp(-i\boldsymbol{\nu}_{ba}\cdot\mathbf{r}_i)\varphi(r_i)|a\rangle^*\langle b|\sum_i\tau_i{}^{(-)}\exp(-i\boldsymbol{\nu}_{ba}\cdot\mathbf{r}_i)\varphi(r_i)(\mathbf{p}_i/m_p)|a\rangle\cdot\hat{\nu}+\mathrm{c.c.}]\\&-(G_A{}^{(\mu)}g_A{}^{(\mu)}-G_P{}^{(\mu)}g_A{}^{(\mu)})[\langle b|\textstyle\sum_i\tau_i{}^{(-)}\exp(-i\boldsymbol{\nu}_{ba}\cdot\mathbf{r}_i)\varphi(r_i)\boldsymbol{\sigma}_i|a\rangle^*\cdot\hat{\nu}\\&\times\langle b|\textstyle\sum_i\tau_i{}^{(-)}\exp(-i\boldsymbol{\nu}_{ba}\cdot\mathbf{r}_i)\varphi(r_i)\boldsymbol{\sigma}_i\cdot(\mathbf{p}_i/m_p)|a\rangle+\mathrm{c.c.}]-(G_A{}^{(\mu)}g_V{}^{(\mu)})\\&\times[\langle b|\textstyle\sum_i\tau_i{}^{(-)}\exp(-i\boldsymbol{\nu}_{ba}\cdot\mathbf{r}_i)\varphi(r_i)\boldsymbol{\sigma}_i|a\rangle^*\cdot\langle b|\sum_i\tau_i{}^{(-)}\exp(-i\boldsymbol{\nu}_{ba}\cdot\mathbf{r}_i)\varphi(r_i)(-i\hat{\nu}\times(\mathbf{p}_i/m_p))|a\rangle+\mathrm{c.c.}]\\&+O(\mathbf{p}_i/m_p)^2\};\end{aligned}\qquad(1)$$

$$G_V{}^{(\mu)}\equiv g_V{}^{(\mu)}(1+\nu_{ba}/2m_p),\qquad g_V{}^{(\mu)}=0.97g_V{}^{(\beta)},$$

$$G_P{}^{(\mu)}\equiv[g_P{}^{(\mu)}-g_A{}^{(\mu)}-g_V{}^{(\mu)}(1+\mu_p-\mu_n)]\nu_{ba}/2m_p,\qquad g_P{}^{(\mu)}=8g_A{}^{(\mu)},$$

$$G_A{}^{(\mu)}\equiv g_A{}^{(\mu)}-g_V{}^{(\mu)}(1+\mu_p-\mu_n)\nu_{ba}/2m_p,\qquad g_A{}^{(\mu)}=1.0g_A{}^{(\beta)},$$

$$(\eta_{ba})^2\equiv(\nu_{ba}/m_\mu)^2(1+\nu_{ba}/Am_p)^{-1}(1+m_\mu/Am_p)^{-3},$$

* Supported in part by the National Science Foundation.

[1] For ${}_1H^2$: J. H. Doede and R. H. Hildebrand, Bull. Am. Phys. Soc. **7**, 490 (1962). For ${}_2He^3$: I. V. Falomkin, A. I. Fillippov, M. M. Kulyukin, B. Pontecorvo, Y. A. Scherbokov, R. M. Sulyaev, V. M. Tsupko-Sitnikov, and O. A. Zaimidoroga, Phys. Letters **1**, 318 (1962); **3**, 229 (1963); **6**, 100 (1963); L. B. Auerbach, R. J. Esterling, R. E. Hill, D. A. Jenkins, J. T. Lach, and N. H. Lipman, Phys. Rev. Letters **11**, 23 (1963). For ${}_2He^4$: R. Bizzarri, E. di Capua, U. Dore, G. C. Gianella, P. Guidoni, and I. Laakso, Phys. Letters **3**, 151 (1962); M. M. Block, T. Kikuchi, D. Koetke, M. Schneeberger, C. R. Sun, R. Walker, G. Culligan, V. L. Telegdi, and R. Winston (to be published).

[2] H. Primakoff, Rev. Mod. Phys. **31**, 802 (1959). We shall refer to this paper as P and use the same notation as developed there (except that the unit vector in the direction of the neutrino momentum will be called $\hat{\nu}$ rather than $\boldsymbol{\nu}_1$). It is to be noted that in the present paper the calculated values of the various muon total capture rates are always averaged over the spin orientation substates of the parent nuclear state $|a\rangle$.

[3] A. Fujii and H. Primakoff, Nuovo Cimento **12**, 327 (1959). We shall refer to this paper as FP and use the same notation as developed there; this notation is basically the same as that in P. A relativistic correction term proportional to $G_A{}^{(\mu)}g_V{}^{(\mu)}$, omitted in Eq. (5b) of FP, is included in Eq. (1); one of us (H.P.) wishes to thank Dr. J. C. Sens for pointing out to him that this term must, in general, be considered and also to thank Dr. L. Wolfenstein for pointing out to him an error in the original version of the argument given with regard to the neglect of certain terms in C_V, C_A [see after Eq. (4) below]. See also J. R. Luyten, H. P. C. Rood, and H. A. Tolhoek, Nucl. Phys. **41**, 236 (1963), who give a general treatment of the relativistic corrections which is similar to that in our Sec. II.

where $|b\rangle$ are the various energetically accessible states of the daughter nucleus and where the last three terms give the relativistic corrections in question. We shall calculate the values of these last three terms, C_V, C_A, C_{VA}, in *closure* approximation, i.e., by summation over *all* states $|b\rangle$ in Eq. (1). We then obtain, neglecting, in addition, terms

$$\approx\left\langle\frac{1}{3}\frac{\partial}{\partial r_i}\ln\varphi(r_i)\right\rangle_a\Big/\langle\nu\rangle_a\approx\tfrac{1}{3}Z_{\text{eff}}/137$$

smaller than the terms kept,

$$C_V=K_V\int\frac{d\hat{\nu}}{4\pi}\left\{\left\langle a\left|\sum_{i,j}{}'\,\tau_i^{(+)}\tau_j^{(-)}\exp(i\langle\nu\rangle_a\hat{\nu}\cdot\mathbf{r}_{ij})\varphi(r_i)\varphi(r_j)\frac{\mathbf{p}_j}{m_p}+\text{H.c.}\right|a\right\rangle\cdot\hat{\nu}\right\}$$

$$=K_V\frac{\langle\nu\rangle_a}{m_p}\{\langle a|\sum_{i,j}{}'\,\tfrac{1}{4}(\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j-\tau_i^{(3)}\tau_j^{(3)})j_0(\langle\nu\rangle_a r_{ij})\varphi(r_i)\varphi(r_j)|a\rangle\}\,,\tag{2}$$

$$K_V\equiv-(G_V^{(\mu)}g_V^{(\mu)})\frac{Z^3}{2\pi^2}\frac{m_\mu^5}{(137)^3}(\langle\eta\rangle_a)^2,$$

$$C_A=K_A\int\frac{d\hat{\nu}}{4\pi}\left\{\left\langle a\left|\sum_{i,j}{}'\,\tau_i^{(+)}\tau_j^{(-)}\exp(i\langle\nu\rangle_a\hat{\nu}\cdot\mathbf{r}_{ij})\varphi(r_i)\varphi(r_{ij})\boldsymbol{\sigma}_i\boldsymbol{\sigma}_j\cdot\frac{\mathbf{p}_j}{m_p}+\text{H.c.}\right|a\right\rangle\cdot\hat{\nu}\right\}$$

$$\cong K_A\frac{\langle\nu\rangle_a}{m_p}\{\langle a|\sum_{i,j}{}'\,\tfrac{1}{4}(\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j-\tau_i^{(3)}\tau_j^{(3)})j_0(\langle\nu\rangle_a r_{ij})\varphi(r_i)\varphi(r_j)\tfrac{1}{3}\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j|a\rangle\}\,,\tag{3}$$

$$K_A\equiv-(G_A^{(\mu)}-G_P^{(\mu)})g_A^{(\mu)}\frac{Z^3}{2\pi^2}\frac{m_\mu^5}{(137)^3}(\langle\eta\rangle_a)^2,$$

$$C_{AV}=K_{AV}\int\frac{d\hat{\nu}}{4\pi}\{\langle a|\sum_{i,j}{}'\,\tau_i^{(+)}\tau_j^{(-)}\exp(i\langle\nu\rangle_a\hat{\nu}\cdot\mathbf{r}_{ij})\varphi(r_i)\varphi(r_j)(i\boldsymbol{\sigma}_i\times\mathbf{p}_j/m_p)+\text{H.c.}|a\rangle\cdot\hat{\nu}\}$$

$$=K_{AV}\frac{\langle\nu\rangle_a}{m_p}\left\{\left\langle a\left|\sum_{i,j}{}'\,\tfrac{1}{4}(\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j-\tau_i^{(3)}\tau_j^{(3)})\frac{j_1(\langle\nu\rangle_a r_{ij})}{(\langle\nu\rangle_a r_{ij})}\varphi(r_i)\varphi(r_j)\cdot[(\mathbf{r}_{ij}\times\mathbf{p}_j)\cdot\boldsymbol{\sigma}_i+(\mathbf{r}_{ji}\times\mathbf{p}_i)\cdot\boldsymbol{\sigma}_j]\right|a\right\rangle\right\},\tag{4}$$

$$K_{AV}\equiv-(G_A^{(\mu)}g_V^{(\mu)})\frac{Z^3}{2\pi^2}\frac{m_\mu^5}{(137)^3}(\langle\eta\rangle_a)^2,$$

where $\langle\nu\rangle_a$, $(\langle\eta\rangle_a)^2$ are suitable averages of ν_{ba}, $(\eta_{ba})^2$ [Eq. (1)], and where the approximate equality in Eq. (3) refers to the neglect of the contribution of d-wave neutrinos. We have also set equal to zero (and so not written out) terms originally appearing in C_V, C_A of the form

$$\langle a|\sum_{i,j}{}'\,\tfrac{1}{4}(\boldsymbol{\tau}_i\times\boldsymbol{\tau}_j)^{(3)}f(\mathbf{r}_i,\mathbf{p}_i,\boldsymbol{\sigma}_i,\mathbf{r}_j,\mathbf{p}_j,\boldsymbol{\sigma}_j)|a\rangle;$$

this follows in the case that $|a\rangle$ is of the type given in Eq. (6) below or in the case that $A-Z=Z$; in the general case these terms are of order $(A-2Z)/Z$ relative to the terms retained and, since we attempt no detailed quantitative discussion of the heavier nuclei, we neglect them.

As regards C_{AV}, Eq. (4) shows that we can write

$$C_{AV}=\sum_{i,j}{}'\langle a|\,\mathbf{U}(\boldsymbol{\tau}_i,\boldsymbol{\tau}_j,\boldsymbol{\sigma}_i)\cdot\mathbf{V}(\mathbf{r}_i,\mathbf{r}_j,\mathbf{p}_j)|a\rangle\,,\tag{5}$$

where $\mathbf{U}$ is vector in spin-space and $\mathbf{V}$ a vector in position-space. Consequently, C_{AV} is of order J_a/A relative to C_V or C_A and so may be neglected for medium-heavy and heavy nuclei.[4] For the very light nuclei: ${}_1H^2$, ${}_2He^3$, ${}_2He^4$,

[4] J_a is the nuclear spin quantum number of the state $|a\rangle$ and never appreciably exceeds unity.

we can, to a high degree of accuracy, factorize $|a\rangle$ in the form,

$$|a\rangle=\chi_a(\cdots,\tau_i,\sigma_i,\tau_j,\sigma_j,\cdots)\Phi_a(\cdots r_{ij}\cdots)\,, \tag{6}$$

so that

$$\langle a|\mathbf{U}(\boldsymbol{\tau}_i,\boldsymbol{\tau}_j,\boldsymbol{\sigma}_i)\cdot\mathbf{V}(\mathbf{r}_i,\mathbf{r}_j,\mathbf{p}_j)|a\rangle=\langle\chi_a|\mathbf{U}(\boldsymbol{\tau}_i,\boldsymbol{\tau}_j,\boldsymbol{\sigma}_i)|\chi_a\rangle\cdot\langle\Phi_a|\mathbf{V}(\mathbf{r}_i,\mathbf{r}_j,\mathbf{p}_j)|\Phi_a\rangle=0\,, \tag{7}$$

since $\mathbf{V}$ is a vector and Φ_a is spherically symmetric in position-space. Thus, to a sufficient approximation, we can write for the very light nuclei: ${}_1H^2$, ${}_2He^3$, ${}_2He^4$, and also for the medium-heavy and heavy nuclei

$$C_V+C_A=-Z^4\frac{(\langle\eta\rangle_a)^2 m_\mu{}^5}{2\pi^2(137)^3}[(g_V{}^{(\beta)})^2+3(g_A{}^{(\beta)})^2]\mathcal{R}\frac{\langle\nu\rangle_a}{m_p}$$

$$\cdot\frac{\langle a|\sum_{i,j}{}'\,\frac{1}{4}(\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j-\tau_i{}^{(3)}\tau_j{}^{(3)})[G_V{}^{(\mu)}g_V{}^{(\mu)}+(G_A{}^{(\mu)}-G_P{}^{(\mu)})g_A{}^{(\mu)}\frac{1}{3}\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j]j_0(\langle\nu\rangle_a r_{ij})\varphi(r_i)\varphi(r_j)|a\rangle}{Z[(G_V{}^{(\mu)})^2+3(G_A{}^{(\mu)})^2+(G_P{}^{(\mu)})^2-2(G_A{}^{(\mu)})(G_P{}^{(\mu)})]}\,, \tag{8}$$

$$\mathcal{R}\equiv\frac{(G_V{}^{(\mu)})^2+3(G_A{}^{(\mu)})^2+(G_P{}^{(\mu)})^2-2(G_A{}^{(\mu)})(G_P{}^{(\mu)})}{(g_V{}^{(\beta)})^2+3(g_A{}^{(\beta)})^2}\,,$$

which is basic in what follows.

We proceed to apply the procedure of Eqs. (4)–(8) of P to Eqs. (1) and (8). A straightforward calculation yields

$$\Lambda^{(\mu)}(a)=Z_{\text{eff}}{}^4(\langle\eta\rangle_a)^2(272\ \text{sec}^{-1})\mathcal{R}\{1-[\mathcal{I}_a-\Delta\mathcal{I}_a]\}\,, \tag{9}$$

where $Z_{\text{eff}}{}^4$, $\mathcal{I}_a$ are defined in Eqs. (4b), (6a)–(8) of P, and where

$$\Delta\mathcal{I}_a\equiv\frac{C_V+C_A}{Z_{\text{eff}}{}^4(\langle\eta\rangle_a)^2(272\ \text{sec}^{-1})\mathcal{R}}$$

$$\cong-\left(\frac{\langle\nu\rangle_a}{m_p}\right)\cdot\frac{\langle a|\sum_{i,j}{}'\,\frac{1}{4}(\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j-\tau_i{}^{(3)}\tau_j{}^{(3)})[G_V{}^{(\mu)}g_V{}^{(\mu)}+(G_A{}^{(\mu)}-G_P{}^{(\mu)})g_A{}^{(\mu)}\frac{1}{3}\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j]j_0(\langle\nu\rangle_a r_{ij})\varphi(r_i)\varphi(r_j)|a\rangle}{Z[(G_V{}^{(\mu)})^2+3(G_A{}^{(\mu)})^2+(G_P{}^{(\mu)})^2-2G_A{}^{(\mu)}G_P{}^{(\mu)}]\left(\int|\varphi(r)|^2\mathfrak{D}_a(r)d\mathbf{r}\right)}$$

$$=x_a\left\{\left[\frac{A}{2Z}-\frac{\langle a|G_V{}^{(\mu)}g_V{}^{(\mu)}[\mathbf{T}^2-(T^{(3)})^2]+\frac{1}{3}(G_P{}^{(\mu)}-G_A{}^{(\mu)})g_A{}^{(\mu)}[(\mathbf{Y}^{(1)})^2+(\mathbf{Y}^{(2)})^2]|a\rangle}{Z[G_V{}^{(\mu)}g_V{}^{(\mu)}+(G_A{}^{(\mu)}-G_P{}^{(\mu)})g_A{}^{(\mu)}]}\right]\alpha_a{}^{(+)}\right.$$

$$\left.+\left[\frac{A-Z}{2}-\frac{[G_V{}^{(\mu)}g_V{}^{(\mu)}-\frac{1}{3}(G_A{}^{(\mu)}-G_P{}^{(\mu)})g_A{}^{(\mu)}]\langle a|\mathbf{S}_p{}^2+\mathbf{S}_n{}^2-(\mathbf{S}_p+\mathbf{S}_n)^2|a\rangle}{Z[G_V{}^{(\mu)}g_V{}^{(\mu)}+(G_A{}^{(\mu)}-G_P{}^{(\mu)})g_A{}^{(\mu)}]}\right]\alpha_a{}^{(-)}\right\}, \tag{10}$$

$$x_a\equiv\left(\frac{\langle\nu\rangle_a}{m_p}\right)\cdot\frac{G_V{}^{(\mu)}g_V{}^{(\mu)}+(G_A{}^{(\mu)}-G_P{}^{(\mu)})g_A{}^{(\mu)}}{(G_V{}^{(\mu)})^2+3(G_A{}^{(\mu)})^2+(G_P{}^{(\mu)})^2-2(G_A{}^{(\mu)})(G_P{}^{(\mu)})}\,,$$

$$\alpha_a{}^{(\pm)}=\frac{\int\int j_0(\langle\nu\rangle_a|\mathbf{r}-\mathbf{r}'|)\varphi(r)\varphi(r')\frac{1}{2}[F_a{}^{(+)}(\mathbf{r},\mathbf{r}')\pm F_a{}^{(-)}(\mathbf{r},\mathbf{r}')]d\mathbf{r}d\mathbf{r}'}{\int|\varphi(r)|^2\mathfrak{D}_a(r)d\mathbf{r}}\,,$$

with $\mathfrak{D}_a(r)$, $F_a{}^{(\pm)}(\mathbf{r},\mathbf{r}')$ defined as in Eqs. (4b), (7b) of P.[5] Equation (10) shows that $\Delta\mathcal{I}_a$, like $\mathcal{I}_a$ [see Eqs. (6c)–(8) of P], is never negative and would vanish if the nucleus had $Z=A$ or if $\langle\nu\rangle_a$ were $\gg\{\langle r_{ij}\rangle_{\text{av}}\}^{-1}$—in fact, we essentially have $\Delta\mathcal{I}_a\cong x_a\mathcal{I}_a$. Thus, in closure approximation, the relativistic corrections affect only the estimate of the inhibitory effect of the exclusion principle on the muon capture process and are, therefore, expected to be the most important when $1-\mathcal{I}_a$ is small, i.e., for muon capture by ${}_2He^4$ or by medium-heavy and heavy nuclei with comparatively large $(A-Z)/2A$ [$\mathcal{I}_a\approx3(A-Z)/2A$—see Eqs. (11a)–(14), (18a) of P].

[5] Actually the $F_a{}^{(\pm)}(\mathbf{r},\mathbf{r}')$ of Eq. (7b) of P differs from the $F_a{}^{(\pm)}(\mathbf{r},\mathbf{r}')$ of Eq. (10) by the replacement of $(G_A{}^{(\mu)})^2+\frac{1}{3}[(G_P{}^{(\mu)})^2-2G_A{}^{(\mu)}G_P{}^{(\mu)}]$ by $\frac{1}{3}(G_A{}^{(\mu)}-G_P{}^{(\mu)})g_A{}^{(\mu)}$ in the $\langle a|\cdots|a\rangle$ of numerator and denominator. However, this difference should not have any important effect on the value of $\alpha_a{}^{(\pm)}$.

III. APPLICATIONS

For muon capture by medium-heavy and heavy nuclei with relatively large Z, A, $(A-Z)/2A$, Eqs. (8), (9), and Sec. 3 of P and Eqs. (8)–(10) yield

$$\begin{gathered}\Lambda^{(\mu)}(a)=Z_{\text{eff}}{}^4(\langle\eta\rangle_a)^2(272\ \text{sec}^{-1})\mathcal{R}\{1-(\tfrac{1}{2}(A/Z)\alpha_a{}^{(+)}+\tfrac{1}{2}(A-Z)\alpha_a{}^{(-)})(1-x_a)\}\,,\\ \langle\nu\rangle_a=0.75m_\mu,\quad (\langle\eta\rangle_a)^2=0.56,\quad \mathcal{R}=1.06,\quad x_a=0.03.\end{gathered}\tag{11}$$

Thus, the relativistic corrections "renormalize" the coefficients $\alpha_a{}^{(+)}$, $\alpha_a{}^{(-)}$ of $\frac{1}{2}(A/Z)$, $\frac{1}{2}(A-Z)$ by approximately 3%; numerically, their relative magnitude is

$$x_a\frac{[\frac{1}{2}(A/Z)\alpha_a{}^{(+)}+\frac{1}{2}(A-Z)\alpha_a{}^{(-)}]}{1-[\frac{1}{2}(A/Z)\alpha_a{}^{(+)}+\frac{1}{2}(A-Z)\alpha_a{}^{(-)}]}\approx x_a\frac{[3(A-Z)/2A]}{1-[3(A-Z)/2A]}\approx 0.1\text{–}0.3\,.$$

However, as seen from the form of the right side of Eq. (11), the presence of the relativistic corrections does not change the essential nature of the predicted dependence of $\Lambda^{(\mu)}(a)$ on Z, A.[6]

For muon capture by $_2\text{He}^3$, Eqs. (8)–(10) and Eqs. (4b), (8), (15)–(17) and Table I of P yield

$$\begin{aligned}&\Lambda^{(\mu)}(a)=Z_{\text{eff}}{}^4(\langle\eta\rangle_a)^2(272\ \text{sec}^{-1})\mathcal{R}\{1-\tfrac{1}{2}(\alpha_a{}^{(+)}+\alpha_a{}^{(-)})(1-x_a)\}\,,\\ &Z_{\text{eff}}{}^4=2^4\times0.98;\quad \langle\nu\rangle_a=0.95m_\mu;\quad (\langle\eta\rangle_a)^2=0.78;\quad \mathcal{R}=1.07;\quad x_a=0.04\\ &(\alpha_a{}^{(+)}+\alpha_a{}^{(-)})=\int\int j_0(\langle\nu\rangle_a|\mathbf{r}-\mathbf{r}'|)F_a{}^{(+)}(\mathbf{r},\mathbf{r}')d\mathbf{r}d\mathbf{r}'\\ &\qquad=\int\int\int j_0(\langle\nu\rangle_a|\mathbf{r}-\mathbf{r}'|)|\Phi_a(\mathbf{r},\mathbf{r}',\mathbf{r}'')|^2\delta(\mathbf{r}+\mathbf{r}'+\mathbf{r}'')d\mathbf{r}d\mathbf{r}'d\mathbf{r}''\\ &\qquad=0.70\,,\end{aligned}\tag{12}$$

where $\Phi_a(\mathbf{r},\mathbf{r}',\mathbf{r}'')$ is the $_2\text{He}^3$ position-space wave function and $\varphi(r)$ has been approximated by unity; the numerical value of $(\alpha_a{}^{(+)}+\alpha_a{}^{(-)})$ is obtained by use of a Gaussian form for Φ_a, analogous to that in Eq. (15) below, which predicts the observed $_2\text{He}^3$ charge distribution radius (1.85×10^{-13} cm [7]). Equation (12) yields for the muon total capture rate of $_2\text{He}^3$ (with an estimated uncertainty of about 10%)

$$\Lambda^{(\mu)}(_2\text{He}^3)=(2.31\times10^3\ \text{sec}^{-1})\times1.02=2.36\times10^3\ \text{sec}^{-1}\,,\tag{13}$$

where $2.31\times10^3\ \text{sec}^{-1}$ is the value of $\Lambda^{(\mu)}(_2\text{He}^3)$ for $x_a=0$. Thus, the relativistic corrections are unimportant numerically for $_2\text{He}^3$ since the corresponding exclusion principle inhibition factor $1-\mathcal{I}_a=1-\frac{1}{2}(\alpha_a{}^{(+)}+\alpha_a{}^{(-)})=0.65$ and is not particularly small. The theoretical value of Eq. (13) is to be compared with a recently obtained experimental value[8] of $(2.14\pm0.18)\times10^3\ \text{sec}^{-1}$.

To consider a situation where the relativistic corrections are important numerically we treat muon capture by $_2\text{He}^4$. In this case Eqs. (8)–(10) and Eqs. (4b), (8), (15)–(17) and Table I of P yield,

$$\begin{aligned}&\Lambda^{(\mu)}(_2\text{He}^4)=Z_{\text{eff}}{}^4(\langle\eta\rangle_a)^2(272\ \text{sec}^{-1})\mathcal{R}\{1-(\alpha_a{}^{(+)}+\alpha_a{}^{(-)})(1-x_a)\}\,,\\ &Z_{\text{eff}}{}^4=2^4\times0.98;\quad \langle\nu\rangle_a=0.72m_\mu;\quad (\langle\eta\rangle_a)^2=0.46;\quad \mathcal{R}=1.05;\quad x_a=0.030\,,\\ &(\alpha_a{}^{(+)}+\alpha_a{}^{(-)})=\int\int j_0(\langle\nu\rangle_a|\mathbf{r}-\mathbf{r}'|)F_a{}^{(+)}(\mathbf{r},\mathbf{r}')d\mathbf{r}d\mathbf{r}'\\ &\qquad=\int\int\int\int j_0(\langle\nu\rangle_a|\mathbf{r}-\mathbf{r}'|)|\Phi_a(\mathbf{r},\mathbf{r}',\mathbf{r}'',\mathbf{r}''')|^2\delta(\mathbf{r}+\mathbf{r}'+\mathbf{r}''+\mathbf{r}''')d\mathbf{r}d\mathbf{r}'d\mathbf{r}''d\mathbf{r}'''\,,\end{aligned}\tag{14}$$

where $\Phi_a(\mathbf{r},\mathbf{r}',\mathbf{r}'',\mathbf{r}''')$ is the $_2\text{He}^4$ position-space wave function and $\varphi(r)$ has again been approximated by unity.

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[6] Sections 2, 3 of P; R. Klein and L. Wolfenstein, Phys. Rev. Letters **9**, 408 (1962); J. R. Luyten, H. P. C. Rood, and H. A. Tolhoek, Ref. 3.

[7] H. Collard and R. Hofstadter, Phys. Rev. **131**, 416 (1963); Phys. Rev. Letters **11**, 132 (1963).

[8] See the third of the Falomkin *et al.* papers in Ref. 1.

The value of

$$\langle\nu\rangle_a=0.72m_\mu\ \{\langle\nu\rangle_a=[m_\mu-\langle E_{\rm excit}(\mathrm{H_1}^4)\rangle-(E_{\rm ground\ state}(\mathrm{H_1}^4)-E_{\rm ground\ state}(\mathrm{He_2}^4))][1-m_\mu/2(m_\mu+4m_p)]$$

[see Eq. (2c) of P]} is based on a theoretical calculation of Caine and Jones[9] for the muon total capture rate in $\mu^-+{}_2\mathrm{He}^4\rightarrow{}_1\mathrm{H}^4+\nu_\mu$ with whose results we are in general agreement. The corresponding value of $(\langle\eta\rangle_a)^2((\langle\eta\rangle_a)^2=(\langle\nu\rangle_a/m_\mu)^2(1+\langle\nu\rangle_a/4m_p)^{-1}(1+m_\mu/4m_p)^{-3}$ [see Eq. (3c) of P]) is 0.46. Taking

$$\Phi_a(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4)=N^{1/2}\exp[-(9/64)\sum_{i,j}{}' r_{ij}^2/2\langle r^2\rangle_a],$$
$$\langle r^2\rangle_a\equiv\int\int\int\int(\mathbf{r}_1)^2|\Phi_a(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4)|^2\delta(\mathbf{r}_1+\mathbf{r}_2+\mathbf{r}_3+\mathbf{r}_4)d\mathbf{r}_1d\mathbf{r}_2d\mathbf{r}_3d\mathbf{r}_4,\tag{15}$$

Bietti,[10] and also ourselves, have found that,

$$(\alpha_a^{(+)}+\alpha_a^{(-)})=\exp[-(4/9)(\langle\nu\rangle_a)^2\langle r^2\rangle_a]\tag{16}$$

which, upon use of $\{\langle r^2\rangle_a\}^{1/2}=\{[{}_2\mathrm{He}^4$ charge distribution radius$]^2-[$proton charge distribution radius$]^2\}^{1/2}=\{[1.68\times10^{-13}\text{ cm}]^2-[0.85\times10^{-13}\text{ cm}]^2\}^{1/2}=1.46\times10^{-13}$ cm, gives[11] (with $\langle\nu\rangle_a=0.72m_\mu$)

$$(\alpha_a^{(+)}+\alpha_a^{(-)})=0.87.\tag{17}$$

Equations (17) and (14) yield for the muon total capture rate of ${}_2\mathrm{He}^4$

$$\Lambda^{(\mu)}({}_2\mathrm{He}^4)=(269\text{ sec}^{-1})\times1.20=324\text{ sec}^{-1},\tag{18}$$

where 269 sec^{-1} is the value of $\Lambda^{(\mu)}({}_2\mathrm{He}^4)$ for $x_a=0$; thus the relativistic corrections produce a 20% increase in the calculated muon total capture rate of ${}_2\mathrm{He}^4$. It must also be noted that Bietti[10] has shown that $(\alpha_a^{(+)}+\alpha_a^{(-)})$ is quite insensitive to the exact functional dependence of $\Phi_a(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4)$ on r_{ij} for a given $\langle\nu\rangle_a$ and $\langle r^2\rangle_a$; hence, the uncertainty in $\Lambda^{(\mu)}({}_2\mathrm{He}^4)$ arises largely from uncertainties in $\langle\nu\rangle_a$ and $\langle r^2\rangle_a$ and may be estimated as ±65 sec^{-1}.[12] We may also mention that if $(g_P^{(\mu)}/g_A^{(\mu)})$ is larger than the value 8 quoted in Eq. (1) and used so far in this paper,[13] e.g., if $(g_P^{(\mu)}/g_A^{(\mu)})$ is taken as 16,[14] then the values of $\mathfrak{R}$ and x_a become 0.98 and 0.025 [Eqs. (8), (10)], and Eqs. (17) and (14) yield

$$\Lambda^{(\mu)}({}_2\mathrm{He}^4)=(250\text{ sec}^{-1})\times1.16=290\text{ sec}^{-1},\tag{19}$$

with an uncertainty ±60 sec^{-1}.[12]

Recently, Bizzarri *et al.*[1] have reported an experimental value for the muon total capture rate in ${}_2\mathrm{He}^4$ which is (450 ± 90) sec^{-1}, while a measurement by Block *et al.*[1] of the same quantity yields (368 ± 47) sec^{-1}. The theoretical value in Eq. (18) agrees with the Block *et al.* value within the over-all uncertainty; the agreement of the theoretical value in Eq. (19) with the Block *et al.* value, while again within the over-all uncertainty, is less good. If future experimental and theoretical values of $\Lambda^{(\mu)}({}_2\mathrm{He}^4)$ (the latter for given $g_V^{(\mu)}$, $g_A^{(\mu)}$, $g_P^{(\mu)}$) can *each* be specified with an uncertainty of no more than 5%, their comparison will determine $g_P^{(\mu)}/g_A^{(\mu)}$ to an accuracy sufficient to test the one-pion-pole dominance hypothesis.[13] It is also clear that a similar situation exists with respect to the comparison of the experimental and theoretical values of $\Lambda^{(\mu)}({}_2\mathrm{He}^3)$.

ACKNOWLEDGMENTS

We are very grateful to Professor Martin M. Block for communication of the results of the experiment of the Northwestern-Chicago group and for helpful discussion and correspondence.

[9] C. A. Caine and P. S. H. Jones, Nucl. Phys. **44**, 177 (1963).

[10] A. Bietti, Nuovo Cimento **20**, 1043 (1961). See also A. Bietti and P. Di Porto, Nuovo Cimento **28**, 270 (1963).

[11] G. R. Burleson and H. W. Kendall, Nucl. Phys. **19**, 68 (1960), give the ${}_2\mathrm{He}^4$ charge distribution radius. For the proton charge distribution radius see G. R. Bishop, *Proceedings of the 1962 International Conference on High-Energy Physics at CERN* (CERN, Geneva, 1962), p. 734.

[12] We consider $(\langle\nu\rangle_a)^2$ and $\langle r^2\rangle_a$ uncertain by 7.5 and 5%, respectively, and note that the $\Lambda^{(\mu)}({}_2\mathrm{He}^4)$ of Eqs. (14)–(19) is, to a good approximation, proportional to $(\langle\nu\rangle_a)^4\langle r^2\rangle_a$.

[13] The value 8 for $(g_P^{(\mu)}/g_A^{(\mu)})$ follows from the assumption that both the pseudoscalar and the axial-vector-divergence nucleon form-factors are dominated (at momentum transfers $<m_\pi$) by the pole associated with the exchange of one pion. See e.g., S. Treiman, *Weak Interactions and Topics in Dispersion Physics*, edited by C. Fronsdal (W. A. Benjamin, Inc., New York, 1963), pp. 104–108.

[14] A value of $(g_P^{(\mu)}/g_A^{(\mu)})$ larger than 8 may be suggested by available data on the muon total capture rate in hydrogen and on the asymmetry of neutron emission following muon capture in sulfur, magnesium, and calcium—see *Proceedings of the 1962 International Conference on High-Energy Physics at CERN* (CERN, Geneva, 1962), pp. 418–423, 424–427, 821–826, 840–842. Data of R. C. Cohen, S. Devons, and A. D. Kanaris on certain muon partial capture rates in oxygen may also support $(g_P^{(\mu)}/g_A^{(\mu)})>8$—see Phys. Rev. Letters **11**, 134 (1963).